Linear Algebra, Winter 2022

List 3

Matrices

A **matrix** is a grid of numbers. The **dimensions** of a matrix are written in the format "$m \times n$", spoken as "m by n", where m is the number of rows and n is the number of columns (write both numbers; do <u>not</u> multiply them).

50. Give the dimensions of the following matrices:

(a) $\begin{bmatrix} -92 & 8 \\ -78 & -67 \end{bmatrix}$ $\boxed{2 \times 2}$

(b) $\begin{bmatrix} -36 \\ 72 \\ -12 \end{bmatrix}$ $\boxed{3 \times 1}$

(c) $\begin{bmatrix} 75 & 89 & 50 \\ -5 & -81 & 34 \end{bmatrix}$ $\boxed{2 \times 3}$

(d) $\begin{bmatrix} -13 & -63 & -5 \\ 0 & -66 & \frac{1}{2} \\ 31 & \frac{5}{22} & \frac{8}{11} \end{bmatrix}$ $\boxed{3 \times 3}$

(e) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\boxed{2 \times 2}$

(f) $\begin{bmatrix} 58 & -65 & 40 & 8 & -1 & 26 \\ -74 & 58 & -92 & -4 & -21 & 74 \end{bmatrix}$ $\boxed{2 \times 6}$

51. Assume A and B are 3×3 matrices, and $\vec{u}$ and $\vec{v}$ are 3×1 column vectors. For each formula below, does it represent a scalar, a vector, a matrix, or nonsense?

(a) $A + B$ $\boxed{\text{matrix}}$

(b) AB $\boxed{\text{matrix}}$

(c) $A + \vec{u}$ $\boxed{\text{nonsense}}$

(d) $A\vec{u}$ $\boxed{\text{vector}}$

(e) $A/\vec{u}$ $\boxed{\text{nonsense}}$

(f) $\vec{v}B$ $\boxed{\text{nonsense}}$

(g) $\vec{v}/B$ $\boxed{\text{nonsense}}$

(h) $A + \vec{u}$ $\boxed{\text{nonsense}}$

(i) $AB\vec{v}$ $\boxed{\text{vector}}$

(j) $(A + B)(\vec{u} + \vec{v})$ $\boxed{\text{vector}}$

(k) $A(\vec{u} \times \vec{v})$ $\boxed{\text{vector}}$

(l) $(\vec{u} \times \vec{v})A$ $\boxed{\text{nonsense}}$

How to multiply matrices: The number in row i and column j of the matrix AB is the dot product of (Row i from matrix A) and (Column j from matrix B).

52. If A is a 2×2 matrix, B is a 3×3 matrix, and C is a 3×2 matrix, which of the following exist?

(a) AA $\boxed{\text{exists}}$ (2×2)

(b) AB $\boxed{\text{doesn't exist}}$

(c) AC $\boxed{\text{doesn't exist}}$

(d) BA $\boxed{\text{doesn't exist}}$

(e) BB $\boxed{\text{exists}}$ (3×3)

(f) BC $\boxed{\text{exists}}$ (3×2)

(g) CA $\boxed{\text{exists}}$ (3×2)

(h) CB $\boxed{\text{doesn't exist}}$

(i) CC $\boxed{\text{doesn't exist}}$

(j) ABC $\boxed{\text{doesn't exist}}$

(k) BCA $\boxed{\text{exists}}$ (3×2)

(ℓ) ACA $\boxed{\text{doesn't exist}}$

53. (a) Calculate $\begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix}$. $\boxed{\begin{bmatrix} 7 & 24 \\ 3 & -72 \end{bmatrix}}$

(b) Calculate $\begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}$. $\boxed{\begin{bmatrix} 3 & 6 \\ 62 & -68 \end{bmatrix}}$

(c) Compare your answers to parts (a) and (b). They are not equal.

54. Compute the following:

(a) $\begin{bmatrix} 1 & 14 & 8 \\ 6 & -1 & 14 \\ 5 & 11 & -3 \end{bmatrix} + \begin{bmatrix} 11 & 2 & 10 \\ -2 & 14 & 8 \\ -2 & 3 & -4 \end{bmatrix} = \boxed{\begin{bmatrix} 12 & 16 & 18 \\ 4 & 13 & 22 \\ 3 & 14 & -7 \end{bmatrix}}$

(b) $\begin{bmatrix} 1 & 14 & 8 \\ 6 & -1 & 14 \\ 5 & 11 & -3 \end{bmatrix} - \begin{bmatrix} 11 & 2 & 10 \\ -2 & 14 & 8 \\ -2 & 3 & -4 \end{bmatrix} = \boxed{\begin{bmatrix} -10 & 12 & -2 \\ 8 & -15 & 6 \\ 7 & 8 & 1 \end{bmatrix}}$

(c) $3\begin{bmatrix} 0 & -4 & 0 \\ -1 & -1 & 3 \\ -2 & 5 & 14 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -12 & 0 \\ -3 & -3 & 9 \\ -6 & 15 & 42 \end{bmatrix}}$

(d) $\frac{1}{6}\begin{bmatrix} 9 & 14 \\ 6 & 10 \end{bmatrix} = \boxed{\begin{bmatrix} 3/2 & 7/3 \\ 1 & 5/3 \end{bmatrix}}$ (e) $\begin{bmatrix} 8 & 5 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \boxed{\begin{bmatrix} 33 \\ -25 \end{bmatrix}}$ (f) $\begin{bmatrix} 9 & -2 \\ 8 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} 6 \\ 46 \end{bmatrix}}$

(g) $\begin{bmatrix} -5 & 5 & 7 \\ -2 & -3 & 2 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 9 \\ 8 \end{bmatrix} = \boxed{\begin{bmatrix} 86 \\ -17 \\ 35 \end{bmatrix}}$

(h) $\begin{bmatrix} 4 & 8 & 0 \\ -3 & 3 & -3 \\ 8 & 5 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix} = \boxed{\begin{bmatrix} 24 \\ -3 \\ -10 \end{bmatrix}}$

55. Compute $\begin{bmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$. $= \boxed{\begin{bmatrix} 2-\sqrt{2} \\ 6\sqrt{2} \\ 2+\sqrt{2} \end{bmatrix}}$

56. Compute the following, if they exist:

(a) $\begin{bmatrix} 9 & -4 \\ -5 & -5 \end{bmatrix}\begin{bmatrix} 8 & 1 \\ 0 & -3 \end{bmatrix} = \boxed{\begin{bmatrix} 72 & 21 \\ -40 & 10 \end{bmatrix}}$

(b) $\begin{bmatrix} 4 & 5 & 22 \\ 8 & -13 & 4 \end{bmatrix}\begin{bmatrix} 19 & 0 & 35 & 6 \\ 0 & 2 & 2 & 6 \\ 9 & 1 & 19 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 274 & 32 & 568 & 32 \\ 188 & -22 & 330 & -34 \end{bmatrix}}$

(c) $\begin{bmatrix} 19 & 0 & 35 & 6 \\ 0 & 2 & 2 & 6 \\ 9 & 1 & 19 & -1 \end{bmatrix}\begin{bmatrix} 4 & 5 & 22 \\ 8 & -13 & 4 \end{bmatrix}$ does not exist

(d) $\begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 & 2 & 7 \\ 3 & -4 & -1 & 1 & 8 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 6 & -3 & 6 & 21 \\ 6 & -4 & -4 & 6 & 30 \end{bmatrix}}$

(e) $\begin{bmatrix} -2 & -4 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 7 & 8 \\ 2 & 8 \end{bmatrix} = \boxed{\begin{bmatrix} 72 & 48 \\ -36 & -24 \end{bmatrix}}$

(f) $\begin{bmatrix} -4 & -3 & -5 \\ 24 & 6 & 29 \end{bmatrix} \begin{bmatrix} 4 & 13 & 0 \\ 2 & -26 & 9 \end{bmatrix}$ $\boxed{\text{does not exist}}$

57. Compute the following:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 2 \\ 3 & -3 \end{bmatrix} = \boxed{\begin{bmatrix} 8 & 2 \\ 3 & -3 \end{bmatrix}}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 14 & 21 \\ -11 & 23 \end{bmatrix} = \boxed{\begin{bmatrix} 14 & 21 \\ -11 & 23 \end{bmatrix}}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 99 & \frac{1}{10} \\ -37 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 99 & \frac{1}{10} \\ -37 & 2 \end{bmatrix}}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & -2 & 1 \\ 5 & 2 & 5 \\ 7 & 4 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -4 & -2 & 1 \\ 5 & 2 & 5 \\ 7 & 4 & 1 \end{bmatrix}}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 15 & 11 & -15 \\ 17 & 10 & -8 \\ 0 & 0 & -13 \end{bmatrix} = \boxed{\begin{bmatrix} 15 & 11 & -15 \\ 17 & 10 & -8 \\ 0 & 0 & -13 \end{bmatrix}}$

(f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 59 & 28 & 58 \\ 61 & 44 & 67 \\ 22 & 39 & 17 \end{bmatrix} = \boxed{\begin{bmatrix} 59 & 28 & 58 \\ 61 & 44 & 67 \\ 22 & 39 & 17 \end{bmatrix}}$

(g) $\begin{bmatrix} 59 & 28 & 58 \\ 61 & 44 & 67 \\ 22 & 39 & 17 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 59 & 28 & 58 \\ 61 & 44 & 67 \\ 22 & 39 & 17 \end{bmatrix}}$

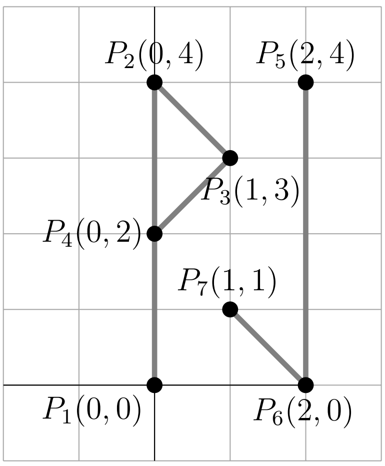
(h) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -39 & -66 & 84 & 66 & -10 \\ -47 & -5 & 17 & -59 & -3 \\ -94 & -90 & -5 & 86 & 31 \\ 25 & 80 & 0 & 35 & 19 \\ -72 & 40 & 99 & 48 & 57 \end{bmatrix} = \boxed{\begin{bmatrix} -39 & -66 & 84 & 66 & -10 \\ -47 & -5 & 17 & -59 & -3 \\ -94 & -90 & -5 & 86 & 31 \\ 25 & 80 & 0 & 35 & 19 \\ -72 & 40 & 99 & 48 & 57 \end{bmatrix}}$

58. For each of the points P_1 through P_7, calculate

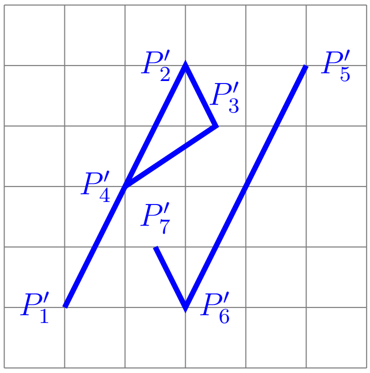
$$P_i' = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} P_i.$$

(For example, for $P_5' = \left[\begin{smallmatrix} 1 & 1/2 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 4 \\ 4 \end{smallmatrix}\right]$.) Plot the points $P_1', \ldots, P_7'$ on a new grid. Connect $P_1' \to P_2' \to P_3' \to P_4'$ with line segments, and connect $P_5' \to P_6' \to P_7'$.

Congratulations. You can write in italics!



$T(P_1) = (0,0)$ $T(P_2) = (2,4)$ $T(P_3) = (5/2,3)$ $T(P_4) = (1,2)$
$T(P_5) = (4,4)$ $T(P_6) = (2,0)$ $T(P_7) = (3/2,1)$



59. If $\begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix} M = \begin{bmatrix} 8 & 25 & 12 \\ 14 & 45 & 22 \end{bmatrix}$, what are the dimensions of matrix M? $\boxed{2 \times 3}$

60. Give the dimensions of the matrix $\begin{bmatrix} 2 & -8 \\ 1 & 5 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 & 11 & 4 \\ -2 & -8 & 6 & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \\ -9 \end{bmatrix} \begin{bmatrix} \frac{2}{7} & 1 & \frac{4}{7} \end{bmatrix}$.

(Do *not* compute the matrix product.)

Dimensions $(3 \times 2)(2 \times 5)(5 \times 1)(1 \times 3)$ leads to $\boxed{3 \times 3}$.

61. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 5 & 2 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$. Write all the products of two matrices from this list that exist (e.g., AA exists, but AC does not).

There are 9 valid products of this form: $\boxed{AA, AB, BA, BB, CD, DC, DE, EA, EB}$

———————Lecture from 6 October covers only to this point.———————

A function f with vector inputs and outputs is a **linear transformation** if

$$f(a\vec{u} + b\vec{v}) = af(\vec{u}) + bf(\vec{v})$$

for all scalars a, b and vectors $\vec{u}, \vec{v}$. Equivalently, $f(a\vec{u} + \vec{v}) = af(\vec{u}) + f(\vec{v})$ is enough, or $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$ and $f(a\vec{u}) = af(\vec{u})$ together.

62. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (x + y, x + 1)$. Show that f is not linear.

There are many examples where the rule for a linear transformation fails for this function. One is that $3\,f(1,1) = 3\,(2,2) = (6,6)$ does not equal $f(3,3) = (6,4)$.

63. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (xy, 1)$. Show that T is not linear.
$T(0,5) + T(1,2) = (0,1) + (2,1) = (2,2)$ does not equal $T(1,7) = (7,1)$.

64. Which of the following are linear transformations?

(a) $g(x, y) = (y, x)$ $\boxed{\text{yes}}$

(b) $L(x, y) = (0, y - 6x)$ $\boxed{\text{yes}}$

(c) $K(x, y) = (6, y - x)$ $\boxed{\text{no}}$

65. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $L(1,1) = (3,-9)$ and $L(2,0) = (6,2)$. Calculate $L(13,3)$. From **Task 26**, $\begin{bmatrix}13\\3\end{bmatrix} = 3\begin{bmatrix}1\\1\end{bmatrix} + 5\begin{bmatrix}2\\0\end{bmatrix}$. Therefore

$$\begin{aligned} L(13,3) &= 3L(1,1) + 5L(6,2) \\ &= 3\begin{bmatrix}3\\-9\end{bmatrix} + 5\begin{bmatrix}6\\2\end{bmatrix} \\ &= [9,-27] + \begin{bmatrix}30\\10\end{bmatrix} \\ &= \boxed{\begin{bmatrix}39\\-17\end{bmatrix}} \end{aligned}$$

66. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $L(1,0) = (4,-1)$ and $L(0,1) = (-3,2)$.

 (a) Find $L(2,-4)$.
 $2L(1,0)+(-4)L(0,1) = 2[4,-1]+(-4)[-3,2] = [8,-2]+[12,-8] = [20,-10]$

 (b) Give a formula for $L(x,y)$ that works for any x and y. $x\begin{bmatrix}4\\-1\end{bmatrix}+y\begin{bmatrix}-3\\2\end{bmatrix}$ or $(4x-3y, -x+2y)$

67. Does there exist a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(0,1) = f(4,3), \quad f(3,0) = f(-3,6), \quad f(1,3) = f(11,11) \ ?$$

$$\boxed{\text{Yes, } f(x,y) = (-x+4y, 2x+3y) = \left[\begin{smallmatrix}-1 & 4\\ 2 & 3\end{smallmatrix}\right]\begin{bmatrix}x\\y\end{bmatrix}}$$

68. Does there exist a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(0,1) = f(4,3), \quad f(0,2) = f(8,6), \quad f(0,3) = f(10,8) \ ?$$

$\boxed{\text{No}}$

69. If $g : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation satisfying

$$g(2,0,0) = (8,8,8), \quad g(0,12,0) = (1,4,-6), \quad g(0,0,5) = (10,0,15),$$

calculate $g(-1,0,6)$. $(-1)\begin{bmatrix}4\\4\\4\end{bmatrix} + 0\begin{bmatrix}1/12\\1/3\\-1/2\end{bmatrix} + 6\begin{bmatrix}2\\0\\3\end{bmatrix} = \boxed{[8,-4,14]}$.

If a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $f(0,1) = (u_1, u_2)$ and $f(0,1) = (w_1, w_2)$, then

$$f(x,y) = \begin{bmatrix}u_1 & w_1\\ u_2 & w_2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

for any (x,y). We call $\left[\begin{smallmatrix}u_1 & w_1\\ u_2 & w_2\end{smallmatrix}\right]$ the "matrix for f" and write M_f for this matrix.

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70. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L(x, y) = (x + y, x)$. Show that L is linear and find the matrix for T.
$L(x_1, y_1) + aL(x_2, y_2) = (x_1 + y_1, x_1) + (ax_2 + ay_2, ay_2)$
$= (x_1 + y_1 + ax_2 + ay_2, x_1 + ay_2) = L(x_1 + ax_2, y_1 + ay_2)$. Matrix: $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$

71. The map $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T_\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

describes counter-clockwise rotation around the origin by an angle α.

Compute $T_{\pi/4}\left(\left[\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}\right]\right) = \boxed{\begin{bmatrix} 3/\sqrt{2} \\ 5/\sqrt{2} \end{bmatrix}}$ and $T_\pi\left(\left[\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}\right]\right) = \boxed{\begin{bmatrix} -4 \\ -1 \end{bmatrix}}$

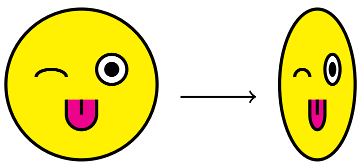
72. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with

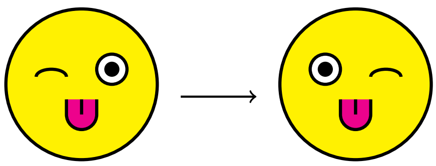
$$f(1, 0, 0) = (5, -1, 4); \qquad f(0, 1, 0) = (2, 1, -7); \qquad f(0, 0, 1) = (3, 2, 4).$$

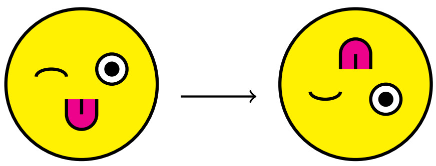
(a) Find $f(2, 2, 5)$. $2(5, -1, 4) + 2(2, 1, -7) + 5(3, 2, 4) = \boxed{(29, 10, 14)}$

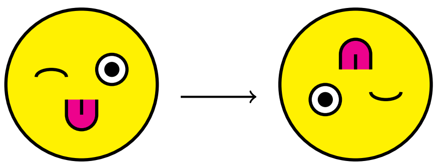
(b) Give the 3×3 matrix for f. $\boxed{M_f = \begin{bmatrix} 5 & 2 & 3 \\ -1 & 1 & 2 \\ 4 & -7 & 4 \end{bmatrix}}$

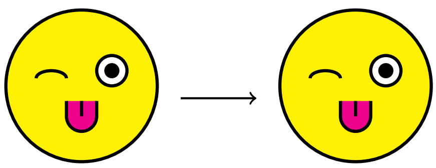
☆73. Match the following linear transformations with their matrices. (That is, which matrix describes (a)? Which matrix describes (b)? And so on.)

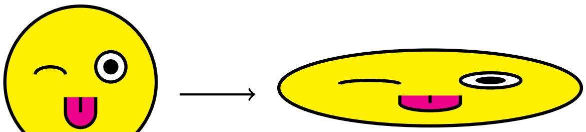
(a)  $M_6 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$

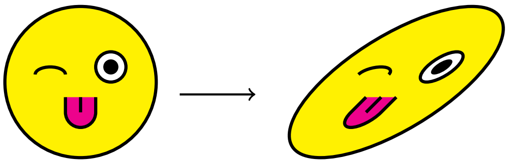
(b)  $M_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(c)  $M_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(d)  $M_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(e)  $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(f)  $M_7 = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$

(g)  $M_5 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Matrices:

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad M_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad M_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$M_5 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad M_6 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \qquad M_7 = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$